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Entropy solutions for some nonlinear elliptic unilateral problems in anisotropic Sobolev spaces

د. محمد علي محسن الحومي
Dr. Mohammed Ali Mohsen Al-Hawmi

Abstract

In this paper, we consider the following nonlinear elliptic unilateral equations of the type

$$-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i(x,u,\nabla u)+g(x,u,\nabla u)+|u|^{p_0-2}u=f \quad in \;\; \Omega$$

In the anisotropic Sobolev space, we prove the existence of entropy solutions for our unilateral problem, the function $g(x,u,\nabla u)$ is a nonlinear lower order term with natural growth with respect to $|\nabla u|$, satisfying the sign condition and the datum f belongs to $L^1(\Omega)$.

key words : Anisotropic Sobolev spaces, nonlinear elliptic unilateral problem, entropy solutions.

1- Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$. Boccardo and Gallouët have considered in [14] the elliptic problem

$$\begin{cases} -div a(x,u,\nabla u)=f & in \quad \Omega \\ u=0 & in \quad \partial\Omega \end{cases} \tag{1.1}$$

where f is a bounded Radon measure on Ω. They have proved the existence of solutions $u \in w_0^{1,q}(\Omega)$ for all $1<q<\overline{q}=\frac{N(p-1)}{N-1}$. Also, they have proved some regularity results. In [11], Boccardo has studied the existence of entropy solutions for the quasilinear elliptic problem of the type

$$\begin{cases} -div a(x,u,\nabla u)=f - div\emptyset(u) & in \quad \Omega \\ u=0 & on \quad \partial\Omega \end{cases} \tag{1.2}$$

with $f\in L^1(\Omega)$ and $\emptyset(u)\in C^0(\mathbb{R},\mathbb{R}^N)$. He has proved the existence and some regularity of solutions. The most studied problems were involved in the p-Laplace operators

$$\Delta_p u = div\,(|\nabla u|^{p-2}\nabla u),$$

with p constant, where the authors have proved the existence of solutions for some nonlinear elliptic unilateral problems in Sobolev spaces. (see for example [13, 14, 20]).

In the framework of Orlicz Sobolev spaces, Aharouch and Bennouna [1] have treated the quasilinear elliptic unilateral problem

$$\begin{cases} -div\, a(x,\nabla u)=f & in \quad \Omega \\ u=0 & on \quad \partial\Omega \end{cases} \tag{1.3}$$

where $f\in L^1(\Omega)$. They have proved the existence and uniqueness of entropy solutions $u \in W_0^1L_M(\Omega)$without any restriction on the N-function M of the Orlicz spaces (i.e. without assuming Δ_2 -condition), we refer also to [5, 10, 16] for more details.

Recently, the anisotropic Sobolev spaces have attracted the attention of many scientists and researchers (see. [7, 18, 22]), this impulse mainly comes from their physical applications in the processes of image restoration, ows of electro-rheological fluids and thermistor problem.

M. AL-hawmi, E.Azroul, H. Hjiaj and A.Touzani have studied in [2] the existence of entropy solutions for some anisotropic quasilinear elliptic unilateral problems

$$\begin{cases} Au = \mu - div\emptyset(u) & in \quad \Omega \\ u = 0 & on \quad \partial\Omega \end{cases} \qquad (1.4)$$

where Au is an operator of Leray-Lions type acting from $W_0^{1,\vec{p}}(\Omega)$ into its dual define by

$$Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u)$$

where $\emptyset(u) \in C^0(\mathbb{R}, \mathbb{R}^N)$ and $\mu = f - div\, F$ such that $f \in L^1(\Omega)$ and $F \in \prod_{i=1}^{N} L^{p'_i}(\Omega)$.

L. M. Kozhevnikova has proved that the entropy solution obtained is a renormalized solution in [19] of the problem under consideration.

$$\begin{cases} \sum_{i=1}^{N} (a_i(x, u, \nabla u))_{x_i} = |u|^{p(x)-2}u + b(x, u, \nabla u) + f(x) & in \quad \Omega \\ u = 0 & on \quad \partial\Omega \end{cases} \qquad (1.5)$$

AL-hawmi, A. Benkirane, H. Hjiaj and A. Touzani have studied in [3] the existence and uniqueness of entropy solutions for some nonlinear elliptic unilateral problems

$$\begin{cases} Au = f & in \quad \Omega \\ u = 0 & on \quad \partial\Omega \end{cases} \qquad (1.6)$$

in Musielak-Orlicz-Sobolev spaces, where $f \in L^1(\Omega)$ and $A : D(A) \subset W_0^1 L_\varphi(\Omega) \longmapsto W_0^1 L_\psi(\Omega)$ is the Leray-Lions operator defined as:

$$Au = -div\, a(x, \nabla u)$$

In this paper, we will prove the existence of entropy solutions for nonlinear anisotropic unilateral elliptic problem of the type

$$\begin{cases} Au + g(x, u, \nabla u) + |u|^{p_0-2}u = f & in \quad \Omega \\ u = 0 & on \quad \partial\Omega \end{cases} \qquad (1.7)$$

where Au is an operator of Leray-Lions type acting from $W_0^{1,\vec{p}}(\Omega)$ into its dual define by

$$Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u)$$

and the function $g(x, u, \nabla u)$ is a non linear lower order term with natural growth with respect to $|\nabla u|$ satisfying the sign condition, that is $g(x, u, \nabla u)\, u \geq 0$.

This paper is organized as follows. In section 2 we recall some de_nitions and basic properties concerning the anisotropic Sobolev spaces.
We introduce in section 3 the assumptions on $a_i(x, u, \nabla u)$ and $g(x, u, \nabla u)$ for which our problem has at least one solution. The section 4 will be devoted to show the existence of entropy solutions for our anisotropic nonlinear elliptic unilateral problem (1.7).

2- Preliminary

Let Ω be an open bounded domain in $\mathbb{R}^N, N \geq 2$ with boundary $\partial\Omega$.
Let $p_0, p_1, \ldots, p_N$ be $N+1$ exponents, with $1 < p_i < \infty$ for $i = 1, \ldots\ldots, N$.
We denote

$$\vec{p} = (p_0, p_1, \ldots\ldots, p_N), \quad D^0 u = u \ \text{ and } \ D^i u = \frac{\partial u}{\partial x_i} \ \text{ for } \ i = 1, \ldots, N$$

We set

$$\underline{p} = min\{p_0, p_1, \ldots, p_N\} \quad \text{then} \quad \underline{p} > 1 . \qquad (2.1)$$

The anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$ is defined as follows

$$W^{1,\vec{p}}(\Omega) = \{ u \in L^{p_0}(\Omega) \ and \ D^i u \in L^{p_i}(\Omega) \ for \ i = 1, 2, \ldots, N\}$$

endowed with the norm

$$||u||_{1,\vec{p}} = \sum_{i=1}^{N} ||D^i u||_{L^{p_i}(\Omega)}. \qquad (2.2)$$

The space $\left(W^{1,\vec{p}}(\Omega), ||u||_{1,\vec{p}}\right)$ is a separable and reflexive Banach space (cf [22]).

We define also $W_0^{1,\vec{p}}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ *in* $W^{1,\vec{p}}(\Omega)$ with respect to the norm (2,2).
Now, we will present some important compacts embedding,

Lemma 2.1 Let Ω be a bounded open set in $\mathbb{R}^N$, then the following embedding are compact

- $if\ \underline{p} < N\ then\ W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \quad \forall q \in [\underline{p}, \underline{p}^*[$, where $\frac{1}{\underline{p}^*} = \frac{1}{\underline{p}} - \frac{1}{N}$,
- $if\ \underline{p} = N\ then\ W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \quad \forall q \in [\underline{p}, \infty[$,
- $if\ \underline{p} > N\ then\ W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap C^0(\overline{\Omega})$.

The proof of this lemma follows from the fact that the embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow W_0^{1,\underline{p}}(\Omega)$ is continuous, and in view of the compact embedding theorem for Sobolev spaces.

Definition 2.1 Let $k > 0$, we consider the truncation function $T_k(.): \mathbb{R} \to \mathbb{R}$ given by

$$T_k(s) = \begin{cases} s & if \quad |s| \le k, \\ k\frac{s}{|s|} & if \quad |s| \ge k, \end{cases}$$

and we define

$$T_0^{1,\vec{p}}(\Omega) := \left\{ u : \Omega \mapsto \mathbb{R} \text{ measurable}, \; T_k(u) \in W_0^{1,\vec{p}}(\Omega) \text{ for any } k > 0 \right\}$$

Proposition 2.1 $u \in T_0^{1,\vec{p}}(\Omega)$, $for\ i = 1, \dots, N$, exists a unique measurable function $v_i : \Omega \mapsto \mathbb{R}$ $such\ that$

$$D^i T_k(u) = v_i x_{\{|u|<k\}} \; a.e.\ in\ \Omega, for\ any\ k > 0,$$

where x_A denotes the characteristic function of a measurable set A. The functions v_i are called the weak partial derivatives of u and are still denoted $D^i u$. Moreover, if u belongs to $W_0^{1,1}(\Omega)$, then v_i coincides with the standard distributional derivatives of u, that is $v_i = D^i u$.
The proof follows the usual techniques developed in [12] for the case of Sobolev spaces. For more details concerning the anisotropic Sobolev spaces, we refer the reader to [8] and [15].

3 - Essential assumptions

Let Ω be a bounded open subset of $\mathbb{R}^N (N \ge 2)$, we assume that the vector $\vec{p} = (p_0, p_1, \dots\dots, p_N)$ satisfying the requirements that $1 < p_i < \infty$ $for\ \ i = 1, 2, \dots, N\}$.
Taking ψ as a measurable function on Ω with values in $\overline{\mathbb{R}}$ such that

$$\psi^+ \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega).$$

We define the convex subset K_ψ by

$$K_\psi = \left\{ v \in W_0^{1,\vec{p}}(\Omega) \text{ such that } v \ge \psi \text{ a.e. in } \Omega \right\}.$$

We consider a Leray-Lions operator $A: W_0^{1,\vec{p}}(\Omega) \mapsto W^{-1,\vec{p}'}(\Omega)$ given by

$$Au = -\sum_{i=1}^{N} D^i a_i(x, u, \nabla u)$$

where $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ are Carathéodory functions, $for\ \ i = 1, 2, \dots, N\}$, (measurable with respect to x in Ω for every (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$ and continuous with respect to (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω), which satisfy the following conditions

$$|a_i(x, s, \xi)| \le \beta(R_i(x) + |s|^{\underline{p}-1} + |\xi|^{p_i - 1}) \quad for\ \ i = 1, 2, \dots, N\}, \qquad (3.1)$$

$$a_i(x, s, \xi)\xi_i \ge \alpha|\xi_i|^{p_i} \quad for\ \ i = 1, 2, \dots, N\}, \qquad (3.2)$$

$$\left(a_i(x, s, \xi) - a_i(x, s, \xi) \right)(\xi_i - \xi_i) > 0 \quad for\ \ \xi_i \ne \xi_i \qquad (3.3)$$

for a.e. x in Ω and all (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$, where $R_i(x)$ is a nonnegative function lying in $L^{p_i'}(\Omega)(\Omega)$ and $\alpha, \beta > 0$.

The nonlinear term $g(x, s, \xi)$ is a Carathéodory function which satisfies

$$g(x, s, \xi)s \ge 0 \qquad (3.4)$$

$$|g(x, s, \xi)| \le b(|s|)(c(x) + \textstyle\sum_{i=1}^{N} |\xi_i|^{p_i}) \qquad (3.5)$$

where $b : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous nondecreasing function, $C \in L^1(\Omega)$ We consider the anisotropic nonlinear elliptic unilateral problem

$$\begin{cases} Au + g(x,u,\nabla u) + |u|^{p_0-2}u = f & in \quad \Omega \\ u = 0 & on \quad \partial\Omega \end{cases} \tag{3.6}$$

Where

$$f \in L^1(\Omega) \tag{3.7}$$

Now, we recall some important Lemmas useful to prove our main result.

Lemma 3.1 (**see [17], Theorem 13.47 page 216**) Let $(u_n)_n$ be a sequence in $L^1(\Omega)$ and $u \in L^1(\Omega)$ such that

(i) $u_n \to u$ $a.e.$ in Ω,

(ii) $u_n \geq 0$ and $u \geq 0$ $a.e.$ in Ω,

(iii) $\int_\Omega u_n\, dx \to \int_\Omega u\, dx$,

then $u_n \to u$ in $L^1(\Omega)$.

Lemma 3.2 (see. [7] page 6) Assuming that (3.1) - (3.3) hold, and let $(u_n)_{n\in N}$ be a sequence in $W_0^{1,\vec{p}}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,\vec{p}}(\Omega)$ and

$$\int_\Omega (|u_n|^{p_0-2}u_n - |u|^{p_0-2}u)(u_n - u)dx + \sum_{i=1}^{N}\int_\Omega (a_i(x,u_n,\nabla u_n) - a_i(x,u_n,\nabla u))(D^i u_n - D^i u)dx \longrightarrow 0, \tag{3.8}$$

then $u_n \to u$ in $W_0^{1,\vec{p}}(\Omega)$ for a subsequence.

4- Main results.

We first give the definition of entropy solutions for our problem (3,6) as follows.

Definition 4.1 A measurable function u is called entropy solution of the anisotropic nonlinear elliptic unilateral problem (3.6) if $T_k(u) \in K_\psi$ and satisfy

$$\begin{cases} \sum_{i=1}^{N}\int_\Omega a_i(x,u,\nabla u)D^i T_k(u_n - v)\,dx + \int_\Omega g(x,u,\nabla u)T_k(u_n - v)dx \\ \qquad \int_\Omega |u|^{p_0-2}uT_k(u_n - v)dx \leq \int_\Omega fT_k(u_n - v)dx \\ \qquad\qquad for\ any\ v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega) \end{cases} \tag{4.1}$$

Our objective is to prove the following existence theorem.

Theorem 4.1 Assuming that (3.1) - (3.5) and (3.7) hold, then the problem (3,6) has at least one entropy solution.

Proof of Theorem 4.1

Step 1: Approximate problems.

Let $(f_n)_{n\in N}$ be a sequence in $W^{-1,\vec{p'}}(\Omega) \cap L^1(\Omega)$ such that $f_n \to f$ in $L^1(\Omega)$ and
$|f_n| \leq |f|$ (for example $f_n = T_n(f)$) =. We consider the approximate problem.

$$\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^i(u_n - v)\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)(u_n - v)dx$$

$$+\int_\Omega |u_n|^{p_0-2}u_n(u_n - v)dx \leq \int_\Omega f_n(u_n - v)dx \qquad (4.2)$$

$$for\ any\ v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$$

for any $v \in K_\psi$, and $g_n(x,s,\xi) = \frac{g(x,s,\xi)}{1+\frac{1}{n}|g(x,s,\xi)|}$, $g_n(x,s,\xi)s \geq 0$,
$|g_n(x,s,\xi)| \leq |g(x,s,\xi)|$ and $|g_n(x,s,\xi)| \leq n$. We define the operators A_n and G_n acted from $W_0^{1,\vec{p}}(\Omega)$ into its dual $W^{-1,\vec{p'}}(\Omega)$ by

$$\langle A_n u, v\rangle = \sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u),\nabla u)D^i v\,dx + \int_\Omega |u|^{p_0-2}uv dx$$

$$\forall\, u, v \in W_0^{1,\vec{p}}(\Omega)$$

and

$$\langle G_n u, v\rangle = \int_\Omega g(x,u,\nabla u)v dx \qquad \forall\, u, v \in W_0^{1,\vec{p}}(\Omega).$$

Lemma 4.1 The operator $B_n = A_n + G_n$ from $W_0^{1,\vec{p}}(\Omega)$ into $W^{-1,\vec{p'}}(\Omega)$ is pseudo-monotone. Moreover, B_n is coercive in the following sense: there exists $v_0 \in K_\psi$ such that

$$\frac{\langle B_n u, v - v_0\rangle}{||u||_{1,\vec{p}}} \to \infty \qquad as\ ||u||_{1,\vec{p}} \to \infty \quad for\ v \in K_\psi.$$

In view of **Lemma 5.1** (**see Appendix**), there exists at least one solution $u_n \in W_0^{1,\vec{p}}(\Omega)$ of the anisotropic nonlinear elliptic unilateral problem (4.2) **(cf. [21], Theorem 8.2 page131)**.

Step 2: A priori estimates.

Let $k \geq max(1, ||\psi^+||_\infty)$; we set $v = u_n - \eta T_k(u_n - \psi^+)\ W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$ for η small enough we have $v \in K_\psi$, thus v is an admissible test function in (4.2), and we obtain

$$\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^i T_k(u_n - \psi^+)\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n - \psi^+)dx$$

$$+\int_\Omega |u_n|^{p_0-2}u_n T_k(u_n - \psi^+)dx \leq \int_\Omega f_n T_k(u_n - \psi^+)dx \qquad (4.3)$$

From (4.3), it follows that

$$\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}} a_i(x,T_n(u_n),\nabla u_n)D^i u_n\,dx + \int_{\{|u_n-\psi^+|\le k\}} g_n(x,u_n,\nabla u_n)u_n dx$$
$$+\int_\Omega |u_n|^{p_0-2}u_nT_k(u_n-\psi^+)dx \le \int_\Omega f_nT_k(u_n-\psi^+)dx \qquad (4.4)$$
$$+\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}} a_i(x,T_n(u_n),\nabla u_n)D^i\psi^+\,dx + \int_{\{|u_n-\psi^+|\le k\}} g_n(x,u_n,\nabla u_n)\psi^+ dx$$

Since $k \ge ||\psi^+||_\infty$ then $T_k(u_n-\psi^+)$ have the same sign as u_n on the set $\{|u_n-\psi^+|>k\}$ it follows that

$$\int_\Omega |u_n|^{p_0-2}u_nT_k(u_n-\psi^+)dx$$
$$=\int_{\{|u_n-\psi^+|\le k\}} |u_n|^{p_0-2}u_n(u_n-v)dx$$
$$+\int_{\{|u_n-\psi^+|>k\}} |u_n|^{p_0-2}u_nT_k(u_n-\psi^+)dx$$
$$\ge \int_{\{|u_n-\psi^+|>k\}} |u_n|^{p_0} - \int_{\{|u_n-\psi^+|\le k\}} |u_n|^{p_0-1}\,|\psi^+|dx$$

In view of (3,2), we obtain

$$\propto \sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}} D^i u_n + \int_{\{|u_n-\psi^+|>k\}} |u_n|^{p_0}$$
$$\le \sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}} a_i(x,T_n(u_n),\nabla u_n)D^i u_n\,dx$$
$$+\int_{\{|u_n-\psi^+|\le k\}} g_n(x,u_n,\nabla u_n)u_n dx$$
$$\le \int_\Omega f_nT_k(u_n-\psi^+)dx \qquad (4.5)$$
$$+\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}} |a_i(x,T_n(u_n),\nabla u_n)||D^i\psi^+|dx$$
$$+\int_{\{|u_n-\psi^+|\le k\}} |g_n(x,u_n,\nabla u_n)||\psi^+|dx$$
$$+\int_{\{|u_n-\psi^+|\le k\}} |u_n|^{p_0-1}\,|\psi^+|dx$$

Thanks to Young's inequality we have

$$\int_{\{|u_n-\psi^+|\le k\}}|u_n|^{p_0-1}\,|\psi^+|dx \le \int_{\{|u_n-\psi^+|\le k\}}|u_n|^{p_0}\,|\psi^+|dx + C_1 \tag{4.6}$$

also, in view of (3,1) and Young's inequality, we have

$$\begin{aligned}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}&|a_i(x,T_n(u_n),\nabla u_n)||D^i\psi^+|\,dx\\ &\le \beta\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}R_i(x)|D^i\psi^+|dx\\ &+\beta\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|u_n|^{\underline{p}-1}|D^i\psi^+\,|dx\\ &+\beta\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}\left|D^iu_n\right|^{p_i-1}|D^i\psi^+|dx \le\ C_2 \qquad (4.7)\\ &+C_3\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|u_n|^{\underline{p}}\,dx+\frac{\alpha}{4}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}\left|D^iu_n\right|^{p_i}dx\\ &+C_4\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}\left|D^i\psi^+\right|^{\underline{p}}dx+C_5\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}\left|D^i\psi^+\right|^{p_i}dx\,,\end{aligned}$$

also, in view of (3.5) and Young's inequality, we have

$$\begin{aligned}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}&|g_n(x,u_n,\nabla u_n)||D^i\psi^+|\,dx\\ &\le\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}b|u_n|(c(x)+\left|D^iu_n\right|^{p_i-1})|D^i\psi^+|dx\\ &\le\ C_6\ +\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}\left|D^iu_n\right|^{p_i-1}|D^i\psi^+|dx \qquad (4.8)\\ &\le\ C_6+\frac{\alpha}{2}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}\left|D^iu_n\right|^{p_i}dx\\ &+C_7\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}\left|D^i\psi^+\right|^{p_i}dx\,,\end{aligned}$$

By combining (4.5) - (4.8), we deduce that

$$\begin{aligned}\frac{\alpha}{4}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}&\left|D^iu_n\right|^{p_i}dx+\frac{1}{2}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|u_n|^{p_0}\,dx \qquad (4.9)\\ &\le k||f||_1+N\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|u_n|^{\underline{p}}\,dx+C_8\end{aligned}$$

Then, there exists a constant C_9 depending only on k such that

$$\frac{\alpha}{4}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}\left|D^i u_n\right|^{p_i} dx \le C_9(k) \qquad (4.10)$$

and since

$$\{x \in \Omega, |u_n| \le k\} \subset \left\{x \in \Omega, |u_n - \psi^+| \le k + ||\psi^+||_\infty\right\}$$

Therefore

$$\sum_{i=1}^{N}\int_{\Omega}\left|D^i T_k(u_n)\right|^{p_i} dx$$

$$= \sum_{i=1}^{N}\int_{\{|u_n|\le k\}}\left|D^i u_n\right|^{p_i} dx + \int_{\Omega}|T_k(u_n)|^{p_0} dx$$

$$\le \sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k+||\psi^+||_\infty\}}\left|D^i u_n\right|^{p_i} dx + |k|^{p_0}|\Omega|$$

$$\le C_{10}\left(k, ||\psi^+||_\infty\right),$$

and we obtain

$$||T_k(u_n)||_{1,\vec{p}} \le C_{11}\left(k, ||\psi^+||_\infty\right),$$

where C_{11} is positive constant that does not depend on n. Thus, the sequence $(T_k(u_n))_n$ is bounded in $W_0^{1,\vec{p}}(\Omega)$uniformly in n, then there exists a subsequence still denoted $(T_k(u_n))_{n\in N}$ and a function $v_k \in W_0^{1,\vec{p}}(\Omega)$ such that

$$\left\{\begin{array}{l} T_k(u_n) \rightharpoonup v_k \quad \text{weakly in } W_0^{1,\vec{p}}(\Omega), \\ T_k(u_n) \rightarrow v_k \quad \text{strongly in } L^{\underline{p}}(\Omega) \text{ and a.e in } \Omega \end{array}\right\} \qquad (4.11)$$

On the other hand, thanks to (4.9) and Poincaré inequality, we have

$$||\nabla T_k(u_n)||_{\underline{p}}^{\underline{p}} = \sum_{i=1}^{N}\int_{\Omega}\left|D^i T_k(u_n)\right|^{\underline{p}} dx \le \sum_{i=1}^{N}\int_{\Omega}\left|D^i T_k(u_n)\right|^{p_i} dx + N|\Omega|$$

$$\le \frac{4}{\alpha}k||f||_1 + \frac{4}{\alpha}N||T_k(u_n)||_{\underline{p}}^{\underline{p}} + C_{12} \qquad (4.12)$$

$$\le \frac{4}{\alpha}k||f||_1 + \frac{4}{\alpha}CN||\nabla T_k(u_n)||_{\underline{p}}^{\underline{p}} + C_{12}$$

Therefore, by taking $\frac{4}{\alpha}CN \le \frac{1}{3}$, there exists a constant C_{13} that does not depend on k and n, such that

$$||\nabla T_k(u_n)||_{\underline{p}} \le C_{13}\, k^{\frac{1}{\underline{p}}} \qquad for \qquad k \ge 1$$

and we obtain

$$k\, meas\,\{|u_n| > k\} = \int_{\{|u_n|>k\}}|T_k(u_n)|dx \le \int_{\Omega}|T_k(u_n)|dx$$

$$\le ||1||_{\underline{p}'}\, ||T_k(u_n)||_{\underline{p}} \qquad (4.13)$$

$$\leq C\|\nabla T_k(u_n)\|_{\underline{p}} \leq C_{14}\, k^{\frac{1}{\underline{p}}},$$

which yields

$$meas\,\{|u_n| > k\} \leq C_{14}\,\frac{1}{k^{1-\frac{1}{\underline{p}}}} \to \infty, \quad k \to \infty \qquad (4.14)$$

Now, we will show that the sequence $(u_n)_n$ is a Cauchy sequence in measure. Indeed, we have for every $\delta > 0$,

$$meas\,\{|u_n - u_m| > \delta\}$$
$$\leq meas\,\{|u_n| > k\} + meas\,\{|u_m| > k\}$$
$$+\ meas\,\{|\,T_k(u_n) - T_k(u_m)| > \delta\}$$

Let $\varepsilon > 0$, in view of (4.14) we may choose $k = k(\varepsilon)$ large enough such that

$$meas\,\{|u_n| > k\} \leq \frac{\varepsilon}{3} \quad and \quad meas\,\{|u_m| > k\} \leq \frac{\varepsilon}{3} \qquad (4.15)$$

Moreover, thanks to (4.11) we have

$$T_k(u_n) \to \eta_k \quad \text{strongly in } L^{\underline{p}}(\Omega) \text{ and a.e in } \Omega$$

Thus $(T_k(u_n))_{n\in N}$ is a Cauchy sequence in measure, and for any $k > 0$ and $\delta, \varepsilon > 0$, there exists $n_0 = n_0(k, \delta, \varepsilon)$ such that

$$meas\,\{|\,T_k(u_n) - T_k(u_m| > \delta\} \leq \frac{\varepsilon}{3} \ \ for\ all\ \ m, n \geq n_0(k, \delta, \varepsilon) \qquad (4.16)$$

By combining (4.15) and (4.16), we conclude that : for all $\delta, \varepsilon > 0$, there exists $n_0 = n_0(\delta, \varepsilon)$ such that

$$meas\,\{|u_n - u_m| > \delta\} \leq \varepsilon \qquad for\ any\ \ m, n \geq n_0.$$

It follows that $(u_n)_n$ is a Cauchy sequence in measure, then converges almost everywhere, for a subsequence, to some measurable function u. Thanks to (4.11) we obtain

$$\left\{ \begin{array}{ll} T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in } W_0^{1,\vec{p}}(\Omega), \\ T_k(u_n) \rightharpoonup T_k(u) & \text{strongly in } L^{\underline{p}}(\Omega) \text{ and a.e in } \Omega \end{array} \right\} \qquad (4.17)$$

Step 3: Convergence of the gradient.

In the sequel, we denote by $\varepsilon_i(n)$ $i = 1,2 \ldots$ a various functions of real numbers which converges to $\mathbf{0}$ as n tends to infinity.

Let $\varphi_k(s) = s.e^{\gamma s^2}$ where $\gamma = \left(\frac{b(k)}{2\alpha}\right)^2$. It is obvious that

$$\varphi'_k(s) - \frac{b(k)}{\alpha}|\varphi_k(s)| \geq \frac{1}{2} \qquad \forall s \in \mathbb{R}.$$

We also consider $h > k > 0$ and $M = 4k + h$ and we set

$$\omega_n = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)).$$

By taking $v = u_n - \eta\varphi_k(\omega_n)$ we have $v \geq \psi$ for η small enough, thus v is an admissible test function in (4.2), and we obtain

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_n(u_n),\nabla u_n)\,\varphi'_k(\omega_n)D^i\omega_n\,dx+\int_{\Omega} g_n(x,u_n,\nabla u_n)\varphi_k(\omega_n)dx$$
$$+\int_{\Omega}|u_n|^{p_0-2}u_n\varphi_k(\omega_n)dx\ \le\int_{\Omega} f_n\varphi_k(\omega_n)dx$$

It is easy to see that $D^i\omega_n=0$ on $\{|u_n|>M\}$ and since

$g_n(x,u_n,\nabla u_n)\varphi_k(\omega_n)\ge 0$ on $\{|u_n|>k\}$, then

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_M(u_n),\nabla T_M(u_n))\,\varphi'_k(\omega_n)D^i\omega_n\,dx+\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\varphi_k(\omega_n)dx$$
$$+\int_{\{|u_n|\le k\}}|u_n|^{p_0-2}u_n\varphi_k(\omega_n)dx\ \le\int_{\Omega} f_n\varphi_k(\omega_n)dx \qquad (4.18)$$

<u>First estimate :</u>
We have

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_M(u_n),\nabla T_M(u_n))\,\varphi'_k(\omega_n)D^i\omega_n\,dx$$
$$=\sum_{i=1}^{N}\int_{\{|u_n|\le k\}} a_i(x,T_M(u_n),\nabla T_M(u_n))\,\varphi'_k(\omega_n)D^iT_{2k}(u_n-T_k(u))\,dx \qquad (4.19)$$
$$+\sum_{i=1}^{N}\int_{\{|u_n|>k\}} a_i(x,T_M(u_n),\nabla T_M(u_n))\,\varphi'_k(\omega_n)D^i\omega_n$$

On one hand, since $|u_n-T_k(u)|\le 2k\ on\ \{|u_n|\le k\}$, then

$$\sum_{i=1}^{N}\int_{\{|u_n|\le k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))\,\varphi'_k(\omega_n)D^iT_{2k}(u_n-T_k(u))\,dx$$
$$=\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n))\,\varphi'_k(\omega_n)(D^iT_k(u_n)-\ D^iT_k(u))\,dx \qquad (4.20)$$
$$+\sum_{i=1}^{N}\int_{\{|u_n|>k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))\,\varphi'_k(\omega_n)D^i\,T_k(u)dx$$

$$Since\ \ 1\le\ \varphi'_k(\omega_n)\le\varphi'_k(2k),\ \ then$$
$$\left|\sum_{i=1}^{N}\int_{\{|u_n|>k\}} a_i(x,T_k(u_n),\nabla T_{Mk}(u_n))\,\varphi'_k(\omega_n)D^i\,T_k(u)\,dx\right|$$
$$\le\varphi'_k(2k)\sum_{i=1}^{N}\int_{\{|u_n|>k\}}|a_i(x,T_k(u_n),\nabla T_k(u_n))||\ D^iT_k(u)|dx$$

and since $\left(\left|a_i(x,T_k(u_n),\nabla T_k(u_n))\right|\right)_{n\in\mathbb{N}}$ is bounded in $L^{p_i'}(\Omega)$, then there exists $\vartheta \in L^{p_i'}(\Omega)$ such that $\left(\left|a_i(x,T_k(u_n),\nabla T_k(u_n))\right|\right) \rightharpoonup \vartheta$ in $L^{p_i'}(\Omega)$.
Therefore

$$\sum_{i=1}^{N}\int_{\{|u_n|>k\}}\left|a_i(x,T_k(u_n),\nabla T_k(u_n))\right|\left|D^iT_k(u)\right|dx$$
$$\longrightarrow \sum_{i=1}^{N}\int_{\{|u_n|>k\}}\vartheta\left|D^iT_k(u)\right|dx = 0$$

It follows that

$$\sum_{i=1}^{N}\int_{\{|u_n|>k\}} a_i(x,T_k(u_n),\nabla T_M(u_n))\,\varphi'_k(\omega_n)D^i\,T_k(u)\,dx = \varepsilon_0(n) \qquad (4.21)$$

On the other hand, for the second term on the right hand side of (4.19), taking $z_n = u_n - T_h(u_n) + T_k(u_n) - T_k(u)$, then

$$\sum_{i=1}^{N}\int_{\{|u_n|>k\}} a_i(x,T_M(u_n),\nabla T_M(u_n))\,\varphi'_k(\omega_n)D^i\omega_n\,dx$$
$$= \sum_{i=1}^{N}\int_{\{|u_n|>k\}\cap\{|z_n|\le 2k\}} a_i(x,T_M(u_n),\nabla T_M(u_n))\,\varphi'_k(\omega_n)D^i\,z_n\,dx$$
$$+\sum_{i=1}^{N}\int_{\{|u_n|>k\}\cap\{|z_n|\le 2k\}} a_i(x,T_M(u_n),\nabla T_M(u_n))\,\varphi'_k(\omega_n)D^i(u_n - T_k(u))\chi_{\{|u_n|>h\}}\,dx$$
$$-\sum_{i=1}^{N}\int_{\{|u_n|>k\}\cap\{|z_n|\le 2k\}} a_i(x,T_M(u_n),\nabla T_M(u_n))\,\varphi'_k(\omega_n)D^i\,T_k(u))\chi_{\{|u_n|\le h\}}\,dx$$
$$\ge -\varphi'_k(2k)\sum_{i=1}^{N}\int_{\{|u_n|>k\}}\left|a_i(x,T_k(u_n),\nabla T_k(u_n))\right|\left|D^iT_k(u)\right|dx$$

In the same way as for (4.21), we can prove that

$$\varphi'_k(2k)\sum_{i=1}^{N}\int_{\{|u_n|>k\}}\left|a_i(x,T_k(u_n),\nabla T_k(u_n))\right|\left|D^iT_k(u)\right|dx = \varepsilon_1(n). \qquad (4.22)$$

After adding (4.19) - (4.22), we obtain

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_M(u_n),\nabla T_M(u_n))\,\varphi'_k(\omega_n)D^i\omega_n\,dx \qquad (4.23)$$
$$\ge \sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n))\,\varphi'_k(\omega_n)(D^iT_k(u_n) - D^iT_k(u))\,dx + \varepsilon_1(n)$$

which is equivalent to say

$$\sum_{i=1}^{N}\int_{\Omega}\left(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\right)\varphi'_k(\omega_n)(D^iT_k(u_n)-D^iT_k(u))\,dx$$
$$\leq\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_M(u_n),\nabla T_M(u_n))\,\varphi'_k(\omega_n)D^i\omega_n\,dx$$
$$-\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_k(u_n),\nabla T_k(u))\,\varphi'_k(\omega_n)(D^iT_k(u_n)-D^iT_k(u))\,dx-\varepsilon_2(n)$$
$$\leq\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_M(u_n),\nabla T_M(u_n))\,\varphi'_k(\omega_n)D^i\omega_n\,dx$$
$$+\varphi'_k(2k)\sum_{i=1}^{N}\int_{\{|u_n|>k\}}|a_i(x,T_k(u_n),\nabla T_k(u_n))||D^iT_k(u)|dx-\varepsilon_2(n)$$

Concerning the second term on the right-hand side of (4.24), by applying the Lebesgue convergence theorem, we have $T_k(u_n)\longrightarrow T_k(u)$ in $L^{p_0}(\Omega)$, then, $|a_i(x,T_k(u_n),\nabla T_k(u))|\longrightarrow|a_i(x,T_k(u),\nabla T_k(u))|$ in $L^{p_i'}(\Omega)$, and since $D^iT_k(u_n)$ converges to $D^iT_k(u)$ weakly in $L^{p_i'}(\Omega)$, we obtain

$$\varepsilon_3(n)=\varphi'_k(2k)\sum_{i=1}^{N}\int_{\{|u_n|>k\}}|a_i(x,T_k(u_n),\nabla T_k(u_n))||D^iT_k(u)|dx\longrightarrow 0 \quad as\quad n\longrightarrow\infty \tag{4.25}$$

We conclude that

$$\sum_{i=1}^{N}\int_{\Omega}\left(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\right)\varphi'_k(\omega_n)(D^iT_k(u_n)-D^iT_k(u))\,dx \tag{4.26}$$
$$\leq\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_M(u_n),\nabla T_M(u_n))\,\varphi'_k(\omega_n)D^i\omega_n\,dx+\varepsilon_3(n)$$

Second estimate :

Now, we treat the second term on the left-hand side of (4.18). From (3.5), we obtain

$$\left|\int_{\{|u_n|\leq k\}}g(x,u_n,\nabla u_n)\varphi_k(\omega_n)dx\right|$$
$$\leq\int_{\{|u_n|\leq k\}}b(|u_n|)\left(c(x)+\sum_{i=1}^{N}|D^iT_k(u_n)|^{p_i}\right)|\varphi_k(\omega_n)|dx$$
$$\leq b(k)\int_{\{|u_n|\leq k\}}c(x)|\varphi_k(\omega_n)|dx$$

$$+\frac{b(k)}{\alpha}\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n))\, D^iT_k(u_n)|\varphi_k(\omega_n)|\,dx$$
$$\le b(k)\int_{\{|u_n|\le k\}} c(x)|\varphi_k(\omega_n)|dx$$
$$+\frac{b(k)}{\alpha}\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n))\, D^iT_k(u)|\varphi_k(\omega_n)|\,dx$$
$$+\frac{b(k)}{\alpha}\sum_{i=1}^{N}\int_{\Omega}\big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\big)(D^iT_k(u_n)-D^iT_k(u))|\varphi_k(\omega_n)|dx$$
$$+\frac{b(k)}{\alpha}\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n)-D^iT_k(u))\,|\varphi_k(\omega_n)|dx$$

Therefore,

$$\frac{b(k)}{\alpha}\sum_{i=1}^{N}\int_{\Omega}\big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\big)(D^iT_k(u_n)-D^iT_k(u))|\varphi_k(\omega_n)|dx$$
$$\ge\left|\int_{\{|u_n|\le k\}} g(x,u_n,\nabla u_n)\varphi_k(\omega_n)dx\right|-b(k)\int_{\{|u_n|\le k\}} c(x)|\varphi_k(\omega_n)|dx$$
$$-\frac{b(k)}{\alpha}\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n)-D^iT_k(u))\,|\varphi_k(\omega_n)|dx$$
$$-\frac{b(k)}{\alpha}\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n))\, D^iT_k(u)|\varphi_k(\omega_n)|\,dx. \qquad (4.27)$$

We have $\varphi_k(\omega_n)\longrightarrow\varphi_k\left(T_{2k}(u-T_h(u))\right)$ weak-$*$ in $L^\infty(\Omega)$ **as** $n\longrightarrow\infty$ as , the

$$\int_{\{|u_n|\le k\}} c(x)|\varphi_k(\omega_n)|dx\longrightarrow\int_{\{|u_n|\le k\}} c(x)\left|\varphi_k\left(T_{2k}(u-T_h(u))\right)\right|dx=0 \qquad (4.28)$$

Concerning the third term on the right-hand side of (4.27), thanks to (4.25), we have

$$\left|\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n)-D^iT_k(u))\,|\varphi_k(\omega_n)|dx\right|$$
$$\le\varphi_k(2k)\sum_{i=1}^{N}\int_{\Omega}|a_i(x,T_k(u_n),\nabla T_k(u))||(D^iT_k(u_n)-D^iT_k(u)|\,dx$$
$$\longrightarrow 0 \text{ as } n\longrightarrow\infty \qquad (4.29)$$

For the last term on the right-hand side of (4.27), we have that $a_i(x,T_k(u_n),\nabla T_k(u_n)))_{n\in\mathbb{N}}$ is bounded in $L^{p_i'}(\Omega)$, then there exists $\psi_i\in L^{p_i'}(\Omega)$ such that $a_i(x,T_k(u_n),\nabla T_k(u_n))\rightharpoonup\psi_i$ in $L^{p_i'}(\Omega)$ and, using the fact that

$$D^i T_k(u)|\varphi_k(\omega_n)| \rightharpoonup D^i T_k(u)\varphi_k\Big(T_{2k}(u - T_h(u))\Big) \quad \text{in } L^{p_i'}(\Omega)$$

it follows that

$$\sum_{i=1}^{N}\int_\Omega a_i(x, T_k(u_n), \nabla T_k(u_n))\, D^i T_k(u)|\varphi_k(\omega_n)|\, dx$$
$$\to \sum_{i=1}^{N}\int_\Omega \psi_i\, D^i T_k(u)\left|\varphi_k\Big(T_{2k}(u - T_h(u))\Big)\right| dx = 0 \qquad (4.30)$$

By combining (4.27) - (4.30), we get

$$\frac{b(k)}{\alpha}\sum_{i=1}^{N}\int_\Omega \Big(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))\Big)\,(D^i T_k(u_n) - D^i T_k(u))|\varphi_k(\omega_n)|dx$$
$$\geq \left|\int_{\{|u_n|\leq k\}} g(x, u_n, \nabla u_n)\varphi_k(\omega_n)dx\right| + \varepsilon_5(n) \qquad (4.31)$$

Third estimate :

We have

$$\int_{\{|u_n|\leq k\}} |u_n|^{p_0-2}u_n\varphi_k(\omega_n)dx$$
$$= \int_\Omega |T_k(u_n)|^{p_0-2}T_k(u_n)(T_k(u_n) - T_k(u))e^{\gamma\omega_n^2}dx$$
$$- \int_{\{|u_n|>k\}} |T_k(u_n)|^{p_0-2}T_k(u_n)(T_k(u_n) - T_k(u))e^{\gamma\omega_n^2}dx$$
$$= \int_\Omega (|T_k(u_n)|^{p_0-2}T_k(u_n) - |T_k(u)|^{p_0-2}T_k(u))(T_k(u_n) - T_k(u))e^{\gamma\omega_n^2}dx$$
$$+ \int_\Omega |T_k(u)|^{p_0-2}T_k(u)(T_k(u_n) - T_k(u))e^{\gamma\omega_n^2}dx$$
$$- \int_{\{|u_n|>k\}} |T_k(u_n)|^{p_0-2}T_k(u_n)(T_k(u_n) - T_k(u))e^{\gamma\omega_n^2}dx$$
$$\geq \int_\Omega (|T_k(u_n)|^{p_0-2}T_k(u_n) - |T_k(u)|^{p_0-2}T_k(u))(T_k(u_n) - T_k(u))dx$$
$$- e^{\gamma(2k)^2}\int_\Omega |T_k(u)|^{p_0-1}|T_k(u_n) - T_k(u)|dx$$
$$- e^{\gamma(2k)^2}\int_{\{|u_n|>k\}} k^{p_0-1}|T_k(u_n) - T_k(u)|dx$$

and as $T_k(u_n) \longrightarrow T_k(u)$ in $L^{p_0}(\Omega)$, then the second and the last term on the right-hand side of the inequality above converges to $\mathbf{0}$ as n goes to infinity. Therefore, we get

$$\int_{\Omega} (|T_k(u_n)|^{p_0-2}T_k(u_n) - |T_k(u)|^{p_0-2}T_k(u))(T_k(u_n) - T_k(u))dx$$
$$\leq \int_{\{|u_n|\leq k\}} |u_n|^{p_0-2}u_n(T_k(u_n) - T_k(u))\varphi_k(\omega_n)dx + \varepsilon_6(n) \quad (4.32)$$

Thanks to (4.26), (4.31) and (4.32), we obtain

$$\frac{1}{2}\sum_{i=1}^{N}\int_{\Omega} \big(a_i(x,T_k(u_n),\nabla T_k(u_n)) - a_i(x,T_k(u_n),\nabla T_k(u))\big)(D^iT_k(u_n) - D^iT_k(u))|\varphi_k(\omega_n)|dx$$
$$+\int_{\Omega} (|T_k(u_n)|^{p_0-2}T_k(u_n) - |T_k(u)|^{p_0-2}T_k(u))(T_k(u_n) - T_k(u))dx$$
$$\leq \sum_{i=1}^{N}\int_{\Omega} a_i(x,T_M(u_n),\nabla T_M(u_n))\,\varphi'_k(\omega_n)D^i\omega_n\,dx$$
$$-\left|\int_{\{|u_n|\leq k\}} g(x,u_n,\nabla u_n)\varphi_k(\omega_n)dx\right|$$
$$+\int_{\{|u_n|\leq k\}} |u_n|^{p_0-2}u_n\varphi_k(\omega_n)dx + \varepsilon_7(n)$$
$$\leq \int_{\Omega} f_n\varphi_k(\omega_n)dx + \varepsilon_7(n) \quad (4.33)$$

We have

$$\int_{\Omega} f_n\varphi_k(\omega_n)dx = \int_{\Omega} f_n\varphi_k(u - T_h(u))dx + \varepsilon_8(n) \quad (4.34)$$

By combining (4.33) and (4.34), we obtain

$$\sum_{i=1}^{N}\int_{\Omega} \big(a_i(x,T_k(u_n),\nabla T_k(u_n)) - a_i(x,T_k(u_n),\nabla T_k(u))\big)(D^iT_k(u_n) - D^iT_k(u))dx$$
$$+\int_{\Omega} (|T_k(u_n)|^{p_0-2}T_k(u_n) - |T_k(u)|^{p_0-2}T_k(u))(T_k(u_n) - T_k(u))dx$$
$$\leq \int_{\Omega} f_n\varphi_k(u - T_h(u))dx + \varepsilon_9(n) \quad (4.35)$$

Then, by letting h and n goes to infinity in (4.35), we get

$$\sum_{i=1}^{N}\int_{\Omega} \big(a_i(x,T_k(u_n),\nabla T_k(u_n)) - a_i(x,T_k(u_n),\nabla T_k(u))\big)(D^iT_k(u_n) - D^iT_k(u))dx$$
$$+\int_{\Omega} (|T_k(u_n)|^{p_0-2}T_k(u_n) - |T_k(u)|^{p_0-2}T_k(u))(T_k(u_n) - T_k(u))dx \longrightarrow 0$$
$$(4.36)$$

Using the **Lemma 3.2**, we deduce that

$$T_k(u_n) \longrightarrow T_k(u) \quad in \quad W_0^{1,\vec{p}}(\Omega) \quad (4.37)$$

Therefore,

$$D^iT_k(u_n) \longrightarrow D^iT_k(u) \quad a.e.\ \text{in}\ \Omega$$

Step 4: The equi-integrability of $g(x,u_n,\nabla u_n)$ and $|u_n|^{p_0-2}u_n$.
In order to pass to the limit in the approximate equation, we show that

$$g(x,u_n,\nabla u_n) \longrightarrow g(x,u,\nabla u) \ \ and \ \ |u_n|^{p_0-2}u_n \longrightarrow |u|^{p_0-2}u \text{ strongly in } L^1(\Omega)$$

By using Vitali's Theorem, it suffices to prove that $g(x,u_n,\nabla u_n)$ and $|u_n|^{p_0-2}u_n$ are uniformly equi-integrable. Indeed, taking $T_1\big(u_n - T_h(u_n)\big)$ as a test function in (4.2), and using (3,2) since $T_1\big(u_n - T_h(u_n)\big)$ have the same sign as u_n, we obtain

$$\begin{aligned}\int_{\{h>|u_n|\}} & g_n(x,u_n,\nabla u_n)\, T_1\big(u_n - T_h(u_n)\big)dx \\ & + \int_{\{h>|u_n|\}} |u_n|^{p_0-2}u_n\, T_1\big(u_n - T_h(u_n)\big)dx \\ & \leq \int_{\{h>|u_n|\}} f_n\, T_1\big(u_n - T_h(u_n)\big)dx \qquad (4.38)\end{aligned}$$

It follows that

$$\begin{aligned}\int_{\{h+1\leq|u_n|\}} & |g_n(x,u_n,\nabla u_n)|dx \\ & + \int_{\{h+1\leq|u_n|\}} |u_n|^{p_0-1}dx \\ & \leq \int_{\{h<|u_n|\}} g_n(x,u_n,\nabla u_n)\, T_1\big(u_n - T_h(u_n)\big)dx \\ & + \int_{\{h<|u_n|\}} |u_n|^{p_0-2}u_n\, T_1\big(u_n - T_h(u_n)\big)dx \\ & \leq \int_{\{h<|u_n|\}} f_n\, T_1\big(u_n - T_h(u_n)\big)dx \leq \int_{\{h<|u_n|\}} |f|dx\end{aligned}$$

thus, for all $\eta > 0$, there exists $h\,(\eta) > 0$ such that

$$\int_{\{h(\eta)\leq|u_n|\}} |g_n(x,u_n,\nabla u_n)|dx + \int_{\{h(\eta)\leq|u_n|\}} |u_n|^{p_0-1}dx \leq \frac{\eta}{2} \qquad (4.39)$$

On the other hand, for any measurable subset $E \subset \Omega$, we have

$$\int_E |g_n(x,u_n,\nabla u_n)|dx + \int_E |u_n|^{p_0-1}dx$$
$$\leq b(h(\eta))\int_E \left(c(x) + \sum_{i=1}^{N} |D^i T_{h(\eta)}(u_n)|^{p_i}\right)dx$$
$$+ \int_E |T_{h(\eta)}(u_n)|^{p_0-1}dx + \int_{\{h(\eta)\leq |u_n|\}} |g_n(x,u_n,\nabla u_n)|dx$$
$$+ \int_{\{h(\eta)\leq |u_n|\}} |u_n|^{p_0-1}dx \tag{4.40}$$

From (4.37), there exists $\beta(\eta) > 0$ such that, for all $E \subseteq \Omega$, with $meas(E) \leq \beta(\eta)$, we have

$$b(h(\eta))\int_E \left(c(x) + \sum_{i=1}^{N} |D^i T_{h(\eta)}(u_n)|^{p_i}\right)dx + \int_E |T_{h(\eta)}(u_n)|^{p_0-1}dx \leq \frac{\eta}{2} \tag{4.41}$$

Finally, by combining (4.39), (4.40) and (4.41), one easily has

$$\int_E |g_n(x,u_n,\nabla u_n)|dx + \int_E |u_n|^{p_0-1}dx \leq \eta \tag{4.42}$$

for all E such that $meas(E) \leq \beta(\eta)$,
we then deduce that $\left(g_n(x,u_n,\nabla u_n)\right)_n$ and $(|u_n|^{p_0-2}u_n)_n$ are equi-integrable, and by Vitali's Theorem we deduce that

$$g(x,u_n,\nabla u_n) \longrightarrow g(x,u,\nabla u) \;\; and \;\; |u_n|^{p_0-2}u_n \longrightarrow |u|^{p_0-2}u \quad \text{in } L^1(\Omega) \tag{4.43}$$

Step 5: Passing to the limit.

Let $v \in K_\psi \cap L^\infty(\Omega)$, by taking $v = u_n - \eta T_k(u_n - \varphi)$ as a test function in (4.2), with η small enough and $\varphi \in K_\psi \cap W_0^{1,\vec{p}}(\Omega)$, and putting $M = k + ||\varphi||_\infty$, we get obtain

$$\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^i T_k(u_n-\varphi)\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n-\varphi)dx$$
$$+ \int_\Omega |u_n|^{p_0-2}u_n T_k(u_n-\varphi)dx \leq \int_\Omega f_n T_k(u_n-\varphi)dx \tag{4.44}$$

On one hand, if $|u_n| > M$ then $|u_n - \varphi| \geq |u_n| - |\varphi| > k$, therefore $\{|u_n - \varphi| \leq k\} \subseteq \{|u_n| \leq M\}$, which implies that

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u_n-\varphi)\,dx$$
$$+\sum_{i=1}^{N}\int_{\Omega} a_i\big(x,T_M(u_n),\nabla T_M(u_n)\big)\,(D^i T_M(u_n)-\ D^i\varphi)\,\chi_{\{|u_n-\varphi|\le k\}}dx$$
$$=\ \sum_{i=1}^{N}\int_{\Omega}\ \Big(a_i\big(x,T_M(u_n),\nabla T_M(u_n)\big)-a_i\big(x,T_M(u_n),\nabla\varphi\big)\Big)\ (D^i T_M(u_n)-\ D^i\varphi)\,\chi_{\{|u_n-\varphi|\le k\}}dx$$
$$+\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_M(u_n),\nabla\varphi)\,(D^i T_M(u_n)-\ D^i\varphi)\,\chi_{\{|u_n-\varphi|\le k\}}dx \qquad (4.45)$$

According to Fatou's Lemma, we obtain

$$\lim_{n\to\infty} inf\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u_n-\varphi)\,dx$$
$$\ge\sum_{i=1}^{N}\int_{\Omega}\ \Big(a_i\big(x,T_M(u_n),\nabla T_M(u_n)\big)$$
$$-\,a_i\big(x,T_M(u_n),\nabla\varphi\big)\Big)\ (D^i T_M(u_n)-\ D^i\varphi)\,\chi_{\{|u_n-\varphi|\le k\}}dx$$
$$+\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_M(u_n),\nabla\varphi)\,(D^i T_M(u_n)$$
$$-\ D^i\varphi)\,\chi_{\{|u_n-\varphi|\le k\}}dx \qquad (4.46)$$

The second term in the right hand side of (4.46) is equal to

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_M(u_n),\nabla\varphi)\,(D^i T_M(u)-\ D^i\varphi)\,\chi_{\{|u_n-\varphi|\le k\}}dx$$

Therefore, we get

$$\lim_{n\to\infty} inf\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u_n-\varphi)\,dx$$
$$\ge\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_M(u),\nabla\varphi)\,(D^i T_M(u)-\ D^i\varphi)\,\chi_{\{|u-\varphi|\le k\}}dx$$
$$=\sum_{i=1}^{N}\int_{\Omega} a_i(x,u,\nabla u)\,(D^i u-\ D^i\varphi)\,\chi_{\{|u_n-\varphi|\le k\}}dx$$
$$=\sum_{i=1}^{N}\int_{\Omega} a_i(x,u,\nabla u)\,D^i T_k(u-\varphi)\,dx$$

On the other hand, we have $T_k(u_n-\varphi)\rightharpoonup T_k(u-\varphi)$ weak-$*$ in $L^\infty(\Omega)$, and in view of (4 .43), We obtain,

$$\int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n-\varphi)dx \longrightarrow \int_\Omega g_n(x,u,\nabla u)T_k(u-\varphi)dx, \qquad (4.47)$$

$$\int_\Omega |u_n|^{p_0-2}u_nT_k(u_n-\varphi)dx \longrightarrow \int_\Omega |u|^{p_0-2}uT_k(u-\varphi)dx \qquad (4.48)$$

$$\text{and} \int_\Omega f_nT_k(u_n-\varphi)dx \longrightarrow \int_\Omega fT_k(u-\varphi)dx \qquad (4.49)$$

Again, since $T_k(u_n-\varphi) \rightharpoonup T_k(u-\varphi)$in $W_0^{1,\vec{p}}(\Omega)$. Which completes the proof of Theorem 4.1. for any positive function $v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$.

5- Appendix

Lemma 5.1 The bounded operator $B_n = A_n + G_n$ acted from $W_0^{1,\vec{p}}(\Omega)$ into $W^{-1,\vec{p}'}(\Omega)$ is pseudo-monotone. Moreover, B_n is coercive in the following sense : there exists $v_0 \in K_\psi$ such that

$$\frac{\langle B_nv, v-v_0\rangle}{||v||_{1,\vec{p}}} \to \infty \qquad as\ ||v||_{1,\vec{p}} \to \infty \quad for\ v \in K_\psi .$$

Proof of Lemma 5.1

Using the $\ddot{Holder's}$ inequality and the growth condition (3.1), we can show that the operator A_n is bounded and by (5.2) we conclude that B_n is bounded. For the coercivity, let $v_0 \in K_\psi$, for any $v \in K_\psi$ we have

$$\begin{aligned}\langle A_nv, v_0\rangle &= \sum_{i=1}^N \int_\Omega a_i(x,T_n(v),\nabla v)D^iv_0\,dx + \int_\Omega |v|^{p_0-2}vv_0dx \\ &\le \sum_{i=1}^N \left(\int_\Omega |a_i(x,T_n(v),\nabla v)|^{p_i'}dx\right)^{\frac{1}{p_i'}} ||D^iv_0||_{p_i} \\ &+ \left(\int_\Omega |v|^{p_0}dx\right)^{\frac{1}{p_0'}} ||v_0||_{p_0} \\ &\le C_{15}\sum_{i=1}^N \left(\int_\Omega \left(R_i(x)^{p_i'} + n^{(\underline{p}-1)p_i'} + \left|D^iv\right|^{p_i}\right)\right)^{\frac{1}{p_i'}} ||v_0||_{1,\vec{p}} \\ &+ \left(\int_\Omega |v|^{p_0}dx\right)^{\frac{1}{p_0'}} ||v_0||_{1,\vec{p}}\end{aligned}$$

and

$$\langle A_nv, v_0\rangle = \sum_{i=1}^N \int_\Omega a_i(x,T_n(v),\nabla v)D^iv\,dx + \int_\Omega |v|^{p_0}dx \ge \alpha' \sum_{i=1}^N \int_\Omega \left|D^iv\right|^{p_i}dx$$

with α' = min α

It follows that

$$\frac{\langle A_n v, v-v_0\rangle}{||v||_{1,\vec{p}}} = \frac{\alpha'}{||v||_{1,\vec{p}}}\sum_{i=1}^{N}\int_\Omega |D^i v|^{p_i}dx - \frac{||v_0||_{1,\vec{p}}}{||v||_{1,\vec{p}}}\left(\int_\Omega |v|^{p_0}dx\right)^{\frac{1}{p'_0}}$$
$$+ C_{15}\frac{||v_0||_{1,\vec{p}}}{||v||_{1,\vec{p}}}\sum_{i=1}^{N}\left(\int_\Omega (R_i(x)^{p'_i} + n^{(\underline{p}-1)p'_i} + |D^i v|^{p_i})\right)^{\frac{1}{p'_i}}||v_0||_{1,\vec{p}}$$
$$+\frac{||v_0||_{1,\vec{p}}}{||v||_{1,\vec{p}}}\left(\int_\Omega |v|^{p_0}dx\right)^{\frac{1}{p'_0}} \tag{5.1}$$

Using the $Hold\ddot{e}r's$ type inequality, we have for all $u, v \in W_0^{1,\vec{p}}(\Omega)$,

$$\langle G_n v, v\rangle = \int_\Omega g_n(x,v,\nabla v)v dx \le ||g_n(x,v,\nabla v)||_{p'_i}||v||_{p_i}$$
$$\le \left(\int_\Omega |g_n(x,v,\nabla v)|^{p'_i}+1)\right)^{\frac{1}{p'_i}}||v||_{1,\vec{p}} \tag{5.2}$$
$$\le \left(n^{p'_i}meas(\Omega)+1\right)^{\frac{1}{p'_i}}||v||_{1,\vec{p}} \le C_{16}||v||_{1,\vec{p}}$$

$$\langle G_n v, v_0\rangle = \int_\Omega g_n(x,v,\nabla v)v_0 dx$$
$$\le \left(\int_\Omega |g(x,v,\nabla v)|^{p'_i}\right)^{\frac{1}{p'_i}}||v_0||_{p_i} \tag{5.3}$$
$$\le C_{17}\sum_{i=1}^{N}\left(\int_\Omega (c(x)^{p'_i} + |D^i v|^{p_i})\right)^{\frac{1}{p'_i}}||v_0||_{1,\vec{p}}$$

It follows that

$$\frac{\langle G_n v, v-v_0\rangle}{||v||_{1,\vec{p}}} = C_{16}\frac{||v_0||_{1,\vec{p}}}{||v||_{1,\vec{p}}} - C_{17}\frac{||v_0||_{1,\vec{p}}}{||v||_{1,\vec{p}}}\sum_{i=1}^{N}\left(\int_\Omega (c(x)^{p'_i} + |D^i v|^{p_i})\right)^{\frac{1}{p'_i}} \tag{5.4}$$

From (5.1) *and* (5.4), as $||v||_{1,\vec{p}} \longrightarrow \infty$, we conclude that

$$\frac{\langle B_n v, v-v_0\rangle}{||v||_{1,\vec{p}}} = \frac{\langle A_n v, v-v_0\rangle}{||v||_{1,\vec{p}}} + \frac{\langle G_n v, v-v_0\rangle}{||v||_{1,\vec{p}}} \longrightarrow \infty \tag{5.5}$$

It remains to show that B_n is pseudo-monotone. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $W_0^{1,\vec{p}}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u & in \quad W^{1,\vec{p}}(\Omega), \\ B_n u_k \rightharpoonup \chi_n & in \quad W^{1,\vec{p}'}(\Omega) \\ \limsup_{k\to\infty} \langle B_n u_k, u_k\rangle \le \langle \chi_n, u\rangle \end{cases} \tag{5.6}$$

We will prove that $\chi_n = B_n u$ and $\langle B_n u_k, u_k\rangle \longrightarrow \langle \chi_n, u\rangle$ as $k \longrightarrow \infty$

Firstly, in view of the compact embedding $W^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega)$ we have $u_k \longrightarrow u$ in $L^{\underline{p}}(\Omega)$ for a subsequence denoted again $(u_k)_{k\in\mathbb{N}}$.

As $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $W^{1,\vec{p}}(\Omega)$, then by the growth condition $\left(a_i(x,T_n(u_k),\nabla u_k)\right)_{k\in\mathbb{N}}$ is bounded in $L^{p_i'}(\Omega)$.Therefore, there exists a function $\varphi_i \in L^{p_i'}(\Omega)$ such that

$$a_i(x,T_n(u_k),\nabla u_k) \rightharpoonup \varphi_i \quad in \quad L^{p_i'}(\Omega) \quad as \quad k\longrightarrow\infty \tag{5.7}$$

and we have

$$|u_k|^{p_0-2}u_k \longrightarrow |u|^{p_0-2}u \quad in \quad L^{p_0'}(\Omega) \quad as \quad k\longrightarrow\infty \tag{5.8}$$

On the one hand, we have for all $v \in W_0^{1,\vec{p}}(\Omega)$
$\langle B_n u_k, u_k\rangle \longrightarrow \langle \chi_n, u\rangle$ as $k\longrightarrow\infty$

$$\begin{aligned}\langle \chi_n, v\rangle &= \lim_{k\to\infty}\langle B_n u_k, u_k\rangle \\ &= \lim_{k\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i v\,dx + \lim_{k\to\infty}\int_\Omega |u_k|^{p_0-2}u_k\,v dx \\ &= \sum_{i=1}^{N}\int_\Omega \varphi_i D^i v\,dx + \int_\Omega |u|^{p_0-2}u\,v dx \end{aligned}\tag{5.9}$$

From relations (5.6) and (5.9), we obtain

$$\begin{aligned}&\lim_{k\to\infty} sup\,\langle B_n u_k, u_k\rangle \\ &= \lim_{k\to\infty} sup\left(\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx + \int_\Omega |u_k|^{p_0}dx\right) \\ &\le \sum_{i=1}^{N}\int_\Omega \varphi_i D^i u\,dx + \int_\Omega |u|^{p_0}dx. \end{aligned}\tag{5.10}$$

On the other hand, by (3.3) we get

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega \left(a_i(x,T_n(u_k),\nabla u_k) - a_i(x,T_n(u_k),\nabla u_k)\right)\left(D^i u_k - D^i u\right)dx \\ &\quad + \lim_{k\to\infty}\int_\Omega (|u_k|^{p_0-2}u_k - |u|^{p_0-2}u)(u_k-u)dx \ge 0 \end{aligned}\tag{5.11}$$

Then

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx \\ &\quad + \int_\Omega |u_k|^{p_0}dx \ge \sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u\,dx \\ &\quad + \int_\Omega |u_k|^{p_0-2}u_k\,u dx + \sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u)\left(D^i u_k - D^i u\right)dx \\ &\quad + \int_\Omega |u|^{p_0-2}u\,(u_k-u)dx \end{aligned}$$

In view of Lebesgue dominated convergence theorem, we have

$T_n(u_k) \longrightarrow T_n(u)$ in $L^{p_i}(\Omega)$ then $a_i(x, T_n(u_k), \nabla u) \longrightarrow a_i(x, T_n(u), \nabla u)$ in $L^{p_i'}(\Omega)$ and by (5.7), we get

$$\liminf_{k\to\infty}\left(\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx + \int_\Omega |u_k|^{p_0}dx\right) \geq \sum_{i=1}^{N}\int_\Omega \varphi_i D^i u\,dx + \int_\Omega |u|^{p_0}dx, \tag{5.12}$$

and

$$\int_\Omega g_n(x,u_k,\nabla u_k)u_k dx \longrightarrow \int_\Omega \phi_i u dx \tag{5.13}$$

Having in mind (5.10) and (5.12) we conclude that

$$\lim_{k\to\infty}\left(\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx + \int_\Omega |u_k|^{p_0}dx\right) = \sum_{i=1}^{N}\int_\Omega \varphi_i D^i u\,dx + \int_\Omega |u|^{p_0}dx, \tag{5.14}$$

According to (5.8), (5.13) and (5.14), we obtain

$$\langle B_n u_k\,, u_k\rangle \longrightarrow \langle \chi_n\,, u\rangle \quad \text{as} \quad k \longrightarrow \infty$$

Now, by (5.14) we can prove that

$$\lim_{k\to\infty}\left(\sum_{i=1}^{N}\int_\Omega \left(a_i(x,T_n(u_k),\nabla u_k) - a_i(x,T_n(u_k),\nabla u_k)\right)\left(D^i u_k - D^i u\right)dx + \int_\Omega (|u_k|^{p_0-2}u_k - |u|^{p_0-2}u)(u_k-u)dx\right) = 0$$

So, by virtue of **Lemma 3.2**, we get

$$u_k \longrightarrow u \;\; in \;\; W_0^{1,\vec{p}}(\Omega) \; and \; D^i u_k \longrightarrow D^i u \;\; a.e. \;\; in \;\; \Omega,$$

then

$$a_i(x,T_n(u_k),\nabla u_k) \longrightarrow a_i(x,T_n(u),\nabla u\;).\;\; in \;\; L^{p_i'}(\Omega) \quad for\; i = 1,\dots,N,$$

and

$$g_n(x,u_k,\nabla u_k) \longrightarrow g_n(x,u,\nabla u) \qquad in \qquad L^{p_0'}(\Omega),$$

which implies that $\chi = B_n u$, which completes the proof of **lemma 5.1.**

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